

A Fast Scalable Algorithm for Discontinuous Optical Flow Estimation

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Abstract—Multiple moving objects, partially occluded objects, or even a single object moving against the background gives rise to discontinuities in the optical flow field in corresponding image sequences. While uniform global regularization based moderately fast techniques cannot provide accurate estimates of the discontinuous flow field, statistical optimization based accurate techniques suffer from excessive solution time. A 'weighted anisotropic' smoothness based numerically robust algorithm is proposed that can generate discontinuous optical flow field with high speed and linear computational complexity. Weighted sum of the first-order spatial derivatives of the flow field is used for regularization. Less regularization is performed where strong gradient information is available. The flow field at any point is interpolated more from those at neighboring points along the weaker intensity gradient-component. Such intensity gradient weighted regularization leads to Euler-Lagrange equations with strong anisotropies coupled with discontinuities in their coefficients. A robust multilevel iterative technique, that recursively generates coarse-level problems based on intensity gradient weighted smoothing weights, is employed to estimate discontinuous optical flow field. Experimental results are presented to demonstrate the efficacy of the proposed technique.

Index Terms—Discontinuous optical flow estimation, weighted anisotropic smoothness, partial differential equation (PDE), scalable algorithm, multilevel iterative methods.

1 INTRODUCTION

DISTRIBUTION of the temporal displacement of intensity patterns in an image sequence is known as the optical flow. The optical flow field and the actual projected 2D velocity match closely at high-gradient image points [11], [29]. High gradient points frequently correspond to locations of 3D motion discontinuities, and consequently give rise to discontinuities in the underlying optical flow field. While considerable attention has been devoted to the accurate estimation of continuous optical flow, fewer attempts have been made in developing computationally efficient techniques for estimating discontinuous optical flow field. A fast scalable algorithm is proposed in this article that is capable of estimating discontinuous optical flow field with reasonable accuracy.

Derivative based optical flow techniques are popular because of their relative speed and ability to produce good qualitative solution. Among the derivative based global techniques, Horn and Schunck employed the uniform first-order smoothness of flow components for regularization. The optical flow field was obtained by solving the corresponding Euler-Lagrange equations by iterative Gauss-

Seidel relaxation [12]. Nagel suggested an oriented smoothness constraint in which smoothness requirement is only imposed orthogonal to the intensity gradient (or principal curvature of local intensity profile) in an attempt to handle occlusion and discontinuity blurring problems of uniform global regularization [17], [18]. Derivative based local techniques, on the other hand, employ a continuous model of the underlying optical flow field in the image neighborhood. The flow field is typically obtained by solving an overdetermined system of intensity (or other gray-level properties) constraint equations [7], [10], [13], [22]. Local techniques are faster and provide accurate solutions when the spatial distribution of intensities satisfies certain condition (e.g., well-conditioned Hessian matrix [29]). However, local methods are usually less stable than the global ones and the computed flow field is sparse.

Several region matching based optical flow techniques have been proposed that attempt to improve the accuracy of optical flow estimation by eliminating the numerical differentiation of image intensities. Anandan proposed one of the earliest region-matching algorithms that employs directional confidences derived from principal curvatures of the matching surface to provide space varying directional weights to the local estimates of the optical flow data [1]. Isotropic smoothness of the flow field is used for regularizing the solution. In Singh's estimation-theoretic framework, the matching errors are converted to a certain idealized probability distribution [23]. He also suggested the use of three image frames to average out spurious minima in the matching error surface due to noise or periodic texture [23].

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Manuscript received Aug. 25, 1994; revised Oct. 17, 1995. Recommended for acceptance by S. Peleg.

For information on obtaining reprints of this article, please send e-mail to: transactions@computer.org, and reference IEEECS Log Number P95168.

The Eigenvalues of the inverse covariance matrix are used as confidence measures to propagate locally computed flow vectors to neighboring points. While both Anandan and Singh employed error surface properties to influence the data constraint term, Zheng and Blostein employed the matching error-weighted regularization to preserve discontinuities [30]. Improved results are obtained in [31], [32] using a multiple off-centered sub-window matching based selective confidence measure to enforce anisotropic regularization [31], [32]. Gauss-Seidel iterations were performed to compute the final flow field. Such pointwise iterative schemes can be very time consuming since error-weighted regularization increases the condition number of the resulting stiffness matrix. Region matching or local correlation-based techniques avoid the numerical computation of intensity derivatives, but employ the second-order properties of the error surface instead, which are prone to numerical inaccuracies as well. While some studies report the robustness of region based methods for integer displacements, accurate subpixel estimates are difficult to obtain from differential properties of the error surface [2]. Also, these are quite sensitive to dilational flows, and performing block matching at every image point is significantly more expensive than computing intensity derivatives.

Sophisticated discontinuity-preserving optical flow techniques have recently been developed based on the Bayesian estimation theory and Markov Random Field (MRF) models. MRF modeling allows us to jointly handle problems of optical flow estimation, issues of motion discontinuity and occlusion processing, and provides a powerful formalism for specifying spatial interactions between features [11]. MRF models were first used by Konrad and Dubois for computing discrete-valued optical flow field [15]. These models incorporate binary 'line processes' to decouple flow estimation processes on either side of a boundary and membrane models [9]. Heitz and Boutheimy presented an MRF based constraint satisfaction framework for estimating multimodal optical flow that uses several complementary constraints [11]. Objective functions associated with MRF models have multiple local minima (essentially due to the presence of both continuous and binary variables), and thus require excessive solution time.

In this paper, we propose a nonuniform regularization based dense optical flow estimation technique, that is fast, robust, and at the same time reasonably preserves discontinuities in the computed optical flow. The proposed technique enforces the weighted anisotropic smoothness of spatial derivatives of the flow field, and attempts to combine the strengths of both local and global methods. This smoothness constraint regularizes the solution more at locations of low 'confidence,' where little information is available in terms of spatial intensity variation. The amount of regularization is small at locations of high 'confidence' with strong intensity gradient, e.g., an edge, to allow for potential motion discontinuities. The relative strength of regularization along the x - and y -axis is dependent on the relative strength of available spatial intensity gradients along these axes. If the intensity gradient-component is stronger along one axis, it is reasonable to allow the data constraint term to play a greater role along that axis, and

thus, the effect of smoothness term is kept small. Also, an intensity gradient-weighted zero-order smoothness (i.e., magnitude penalization) of the flow field itself is introduced to reduce the propagation of flow vectors in weakly textured background points.

The proposed objective function has a unique global minima, and the resulting Euler-Lagrange equations are second-order elliptic in nature. However, the intensity gradient-weighted regularization introduces strong discontinuities coupled with anisotropies in the coefficients of the Euler-Lagrange equations, and subsequently on discretization yields an extremely ill-conditioned system of linear equations. Pointwise iterative methods, e.g., Gauss-Seidel are extremely slow to generate acceptable solutions for such systems. The rate of convergence of Gauss-Seidel depends on the square of the size of discretization, $\sqrt{N}$, where N is the number of image pixels. Thus, Gauss-Seidel relaxation has a time complexity of $O(N^2)$. Moreover, the rate of convergence is adversely dependent on the ratio of maximum to minimum coefficients of the Euler-Lagrange equations, which can be very large for the weighted anisotropic regularization (also for error-weighted regularization in [30]). Therefore, Gauss-Seidel is not suitable for generating discontinuous optical flow field with high speed. Multigrid methods, on the other hand, employ a hierarchy of coarse-level problems to achieve size-independent rate of convergence for second-order elliptic problems; however the convergence of conventional multigrid methods too is dependent adversely on the coefficients of the Euler-Lagrange equations. Conventional averaging based restriction and bilinear interpolation based prolongation operators cannot ensure high rate of convergence for weighted anisotropic regularization based optical flow computation. A multilevel solution strategy is employed in this paper that utilizes the information about the regularization weights in creating restriction and prolongation operators such that an image size-independent high rate of convergence is guaranteed for discontinuous optical flow computation. This solution strategy is based on the concept of *smoothed aggregation*, introduced recently in [26] and analyzed in [27]. Note that for isotropic regularization, such coarsening results in coarse grids with successive three-to-one reduction in both dimensions thereby offering a much lower computational complexity over two to one reduction in conventional algorithms, reported in [20]. In addition, this strategy is entirely algebraic (contrary to geometric) in nature and so suitable for constructing multiresolution pyramid of images, obtained by unstructured sensors (e.g., log-polar).

The rest of this paper is organized as follows. Section 2 deals with the proposed objective function for discontinuous optical flow computation that employs a weighted anisotropic smoothness constraint for regularization. The multigrid based scalable minimization technique is presented in Section 3. Experimental results are presented in Section 4 with several synthetic and real images to demonstrate the effectiveness of the proposed optical flow technique in terms of both accuracy and speed. Conclusions and future research issues are discussed in Section 5.

2 REGULARIZATION FOR DISCONTINUOUS OPTICAL FLOW ESTIMATION

Optical flow computation is a well-known ill-posed problem where available spatiotemporal information does not constrain the solution sufficiently. Therefore, smoothness constraint based regularization is performed to compute the optical flow. In this section, a weighted anisotropic smoothness based regularization technique is proposed that provides relatively accurate estimates of discontinuous optical flow field from a two-frame sequence of images. Such regularization based discontinuous optical flow computation technique results in a strictly convex variational problem, that allows the application of multilevel preconditioner based iterative methods for very fast and robust numerical solution.

2.1 Weighted Anisotropic Smoothness Term

Global regularization based techniques compute the optical flow field by minimizing a quadratic functional. This functional, typically, is a sum of data constraint term and a smoothness term. The data constraint term arises from the conservation of intensity $E(x, y)$ of a moving point $(x(t), y(t))$ over two frames, i.e.,

$$\begin{aligned} \frac{d}{dt} E(x(t), y(t), t) &= 0 \\ \text{or, } \frac{\partial E}{\partial x} \frac{dx(t)}{dt} + \frac{\partial E}{\partial y} \frac{dy(t)}{dt} + \frac{\partial E}{\partial t} &= 0 \\ \text{or, } E_x u + E_y v + E_t &= 0 \end{aligned} \quad (1)$$

where, (u, v) is the optical flow field at point (x, y) . This data constraint term has been used extensively in the optical flow literature. Smoothness based global constraints are used in practice. One such regularization is the Horn and Schunck's first-order smoothness of the optical flow field,

$$\sum_{sm} \lambda [u_x^2 + u_y^2 + v_x^2 + v_y^2] \quad (2)$$

The optical flow is computed by minimizing the objective function,

$$F = \int_{\Omega} (E_x u + E_y v + E_t)^2 d\Omega + \int_{\Omega} \sum_{sm} d\Omega \quad (3)$$

with respect to u, v . The first-order spatial variations of optical flow components are penalized uniformly by a constant, λ at every image point. Such uniform smoothness term penalizes the formation of sharp motion discontinuities and tends to eliminate actual motion discontinuities that may be present in a scene with multiple moving objects, occluded objects, or even a single object moving against the background. Also, the accuracy of optical flow components suffers due to uniform smoothing at points where the intensity variation is primarily in one spatial direction. However, isotropic uniform regularization mechanism is effective in propagating the optical flow field to regions of low texture or intensity gradient.

Local techniques, on the other hand, cannot propagate flow field to uniform regions, but are capable of providing accurate estimates of optical flow vectors if local intensity

profiles obey certain conditions. In image regions where intensity is varying only in one direction, local techniques, in general, compute the normal component of the flow. One attractive attribute of local methods is that they provide the so-called 'confidence' measure at image points to denote the availability of spatial information, and the accuracy of flow estimates. Consider the optical flow functional without the regularization term,

$$\mathcal{F}_{data} = \int_{\Omega_i} (E_x u + E_y v + E_t)^2 d\Omega_i$$

in a local image neighborhood Ω_i . Minimization of this functional with respect to (u, v) yields,

$$\begin{bmatrix} \sum_{\Omega_i} E_x^2 & \sum_{\Omega_i} E_x E_y \\ \sum_{\Omega_i} E_x E_y & \sum_{\Omega_i} E_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum_{\Omega_i} E_x E_t \\ \sum_{\Omega_i} E_y E_t \end{bmatrix} \quad (4)$$

Although strictly speaking, the accuracy of the computed flow or the 'confidence' depends on the condition number of the above matrix, experiments suggest that the sum of eigenvalues [22] (alternatively, the minimum eigenvalue [2]) is a reliable measure of confidence. If the sum is large, the normal velocity can be obtained with high confidence. Otherwise, no flow can be obtained. If there is only one image point $(x, y) \in \Omega_i$, the eigenvalues are given by

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = E_x^2 + E_y^2.$$

If both E_x and E_y are zeros, infinitely many solutions exist for (u, v) . If $|E_x| \gg 0$ and $E_y = 0$, u can be calculated as $-E_t/E_x$, and v cannot be recovered uniquely. If $|E_y| \gg 0$ and $E_x = 0$, v can be obtained as $-E_t/E_y$, but u cannot be recovered uniquely. Thus, the optical flow field (u, v) can be partly recovered from the data constraint term alone.

Our approach to discontinuous optical flow estimation incorporates the strengths of global and local methods. In order to obtain optical flow field with a high accuracy, it is necessary to apply data-weighted smoothness of flow field along different axes. $\lambda_1 + \lambda_2$ provides a reliable measure of the goodness of data. If the sum of eigenvalues is small at some image point, high amount of regularization should be performed to propagate the flow vectors to that point from neighboring points. Otherwise, the data constraint should determine the flow field and regularization weights should be kept small. Relative strength of smoothness along the x - and y -axis should be such that less regularization is performed if the intensity is varying strongly along that axis, i.e., the confidence of data along the axis is high. For example, if $|E_x| \gg 0$ and $E_y = 0$ at any point, regularization should be performed at that point along the y -axis to ensure that the optical flow component v is strongly propagated from neighboring points, while the flow component u is essentially determined by the data constraint term (oriented smoothness based regularization in [18] allows for equal smoothing). Such a regularization term can be written as,

$$\sum_{psm} = \frac{\lambda}{E_x^2 + E_y^2} \left[E_y \left(u_x^2 + v_x^2 \right) + E_x \left(u_y^2 + v_y^2 \right) \right] \quad (5)$$

$$= \alpha(x, y) \left(u_x^2 + v_x^2 \right) + \beta(x, y) \left(u_y^2 + v_y^2 \right)$$

where,

$$\alpha(x, y) = \lambda \cdot \frac{|E_y|}{E_x^2 + E_y^2}$$

$$\beta(x, y) = \lambda \cdot \frac{|E_x|}{E_x^2 + E_y^2}$$

Note that, this smoothness term is anisotropic in nature coupled with discontinuity. It is obvious that if $|E_y| \gg |E_x|$, then the variation of the solution (u, v) is penalized less in y -than in x -direction, and *vice versa*. Thus, the data constraint term plays a stronger role along the y -axis in obtaining the solution. More importantly, deviation from the smoothness of the flow field is penalized less at locations of strong image gradient to allow for optical flow discontinuities at these locations. High intensity gradients correspond to depth gradients or 3D edges that potentially are locations of 3D motion discontinuities. If the gradient magnitude $E_x^2 + E_y^2 \approx 0$, high amount of isotropic smoothness is imposed to propagate solutions from nearby points. If $|E_x| = |E_y|$ at every image point, the proposed smoothness term becomes isotropic, and equivalent to that used in Horn and Schunck's technique.

Global optimization of (5) along with the data constraint term yields more robust flow fields than those obtained by local optimization owing to the fact that the former is able to propagate flow vectors to the points with weak gradient (texture) from neighboring points with stronger gradient. At the same time, by enforcing local smoothness of the flow field, the objective function propagates undesirable flow vectors to background points to some extent. In typical application domains, e.g., navigation or target tracking, object points frequently have more texture than background points. For such applications, the objective function (5) can be modified to penalize the magnitude of flow vectors at points with little texture and temporal gradient. This zero-order penalization can be expressed as $\gamma(\alpha(x, y) + \beta(x, y))$ ($u^2 + v^2$) and the modified objective function as,

$$\mathcal{F}_{pr} = \int \left(E_x u + E_y v + E_t \right)^2 + \gamma(\alpha(x, y) + \beta(x, y)) \left(u^2 + v^2 \right) + \left[\alpha(x, y) \left(u_x^2 + v_x^2 \right) + \beta(x, y) \left(u_y^2 + v_y^2 \right) \right] d\Omega \quad (6)$$

Zero-order penalization contradicts the first-order smoothness of flow field in the sense that at low gradient points, first-order smoothness tries to propagate flow field from neighboring points, whereas zero-order smoothness tries to set zero optical flow there. Its influence is nominal at object points where some spatial gradient is present, i.e., the data constraint $E_x u + E_y v + E_t$ is not identically zero for all (u, v) , since

$$E_x^2 \text{ or } E_y^2 \gg \gamma(\alpha(x, y) + \beta(x, y)) \text{ for } \gamma \ll 1.$$

On the other hand, at background points, where E_x^2 or E_y^2 is very small, $\gamma(\alpha(x, y) + \beta(x, y))$ suppresses the propagation of undesirable flow vectors. Good results are obtained in all our experiments for γ between 0.05% and 0.1%. From the numerical standpoint, zero-order smoothness adds diagonal dominance to the corresponding stiffness matrix arising from finite-difference discretization, and expedites the convergence of iterative solution methods. It is particularly suitable in infra-red ATR (automatic target recognition) scenarios with multiple bright targets moving against a dark background. Note that the zero-order smoothness does not adversely affect the smoothness of flow field at isolated object points with little texture as long as some spatiotemporal information is available at neighboring points. This behavior is in accordance with visual perception in humans (consider a rotating circular disk painted with texture. If uniform texture is present at only a few points, we are able to interpolate the motion from neighboring points, but we perceive a moving disk with several holes, if large areas of uniform texture are present). A similar zero-order smoothness term is mentioned in [3] in the context of shape from shading.

The first-order smoothness term, $\sum_{psm}$ is anisotropic in nature—the deviation of flow vectors from smoothness is penalized differently along the x - and the y -axis. Thus, it is not rotationally invariant contrary to many well-known forms of regularization [24], including the Nagel-Enkelmann's [17]. However, the anisotropic regularization is indeed capable of yielding reasonable estimates of discontinuous flow field when the motion discontinuity is oriented in arbitrary direction. Consider a moving intensity edge that is diagonally oriented. In the proposed technique, same amount of smoothing is imposed at edge points along the x - and y -axis rather than smoothing parallel to the edge. But the scaling factor, $E_x^2 + E_y^2$ in the regularization term assures that the amount of smoothing is much smaller at edge points compared to neighboring non-edge points, since $\sqrt{E_x^2 + E_y^2}$ is larger at edge points. Thus, in a region surrounding the edge, edge points are connected very weakly to their four neighboring non-edge points, whereas non-edge points are strongly connected to one another, thereby allowing for discontinuities to be formed in the computed optical flow. Directional selectivity in human brain has been confirmed in several vision psychology experiments [6]. The scaled matching error measure, used in [30], is rotationally variant too unless the error surface itself is rotationally invariant. The ability to preserve discontinuity in the computed flow is dependent on the scaled matching error which tends to grow near motion discontinuity. We found the matching error to be quite sensitive to the range of underlying motion as well as to the intensity distribution in the neighborhood around discontinuities. It is easier to understand the behavior of E_x and E_y around an edge rather than that of the matching error surface. Also, the fact that a large motion results in large matching errors with better discontinuity preservation

property, imposes a minimum limit on the search window size thereby increasing the computational expense.

The proposed weighted anisotropic smoothness term results in Euler-Lagrange equations that are second-order elliptic in nature. Finite-difference discretization of these Euler-Lagrange equations results in symmetric positive definite stiffness matrix and allows the application of multi-level preconditioner based very fast numerical solution techniques. On the other hand, oriented smoothness based regularization may lead to parabolic equations which are prone to numerical difficulties. The oriented smoothness term proposed in [18] can be expressed as,

$$\sum_{sm} = \frac{\lambda}{E_x^2 + E_y^2 + 2\delta} \begin{bmatrix} (E_y u_x - E_x u_y)^2 \\ + (E_y v_x - E_x v_y)^2 \\ + \delta(u_x^2 + u_y^2 + v_x^2 + v_y^2) \end{bmatrix} \quad (7)$$

Without the loss of generality, assume $E_x^2 + E_y^2 \neq 0$ at any image point so that the isotropic part of smoothness constraint can be excluded. Without the isotropic part, (7) can be written as

$$\sum_{sm} = \frac{\lambda}{E_x^2 + E_y^2} \begin{bmatrix} (E_y u_x - E_x u_y)^2 \\ + (E_y v_x - E_x v_y)^2 \end{bmatrix}$$

Minimization of the data constrain term plus this smoothness term leads to Euler-Lagrange equations:

$$\lambda \cdot \left[\frac{E_y^2}{E_x^2 + E_y^2} u_{xx} + \frac{E_x^2}{E_x^2 + E_y^2} u_{yy} - \frac{2E_x E_y}{E_x^2 + E_y^2} u_{xy} \right] + E_x^2 u + E_x E_y uv = -E_x E_t$$

$$\lambda \cdot \left[\frac{E_y^2}{E_x^2 + E_y^2} v_{xx} + \frac{E_x^2}{E_x^2 + E_y^2} v_{yy} - \frac{2E_x E_y}{E_x^2 + E_y^2} v_{xy} \right] + E_y^2 v + E_x E_y u = -E_y E_t$$

Clearly these are parabolic in nature. Finite-difference discretization is applicable if the intensity is one dimensional, i.e., $E_x E_y = 0$, so that the mixed term u_{xy} is not present. In the general case, finite-difference discretization gives rise to a small negative eigenvalue in the stiffness matrix and the discrete system exhibits a special type of hyperbolic behavior. This leads to the initial convergence of Gauss-Seidel, followed by divergence in later iterations, as observed in some experiments. A nonzero δ introduces ellipticity to the Euler-Lagrange equations and reduces the burden of solution process, but at the same time the effect of oriented smoothness is sacrificed in the actual flow computation.

The weighted anisotropic smoothness and the oriented smoothness measures are derived according to different criteria, and their behaviors are quite different. Even for one-dimensional intensity patterns (no mixed derivative in the oriented smoothness term), the oriented and the proposed smoothness measures differ in that the former smooths both flow components equally, whereas the latter smooths one component of the flow more than the other. This reduces the error encountered in the flow estimation from intensity patterns varying primarily in one direction.

While the proposed smoothness term is dependent directly on the image intensity values and only indirectly on their possible 3D interpretation (e.g., presence of high intensity gradient and possible 3D motion discontinuity), oriented smoothness is more dependent on actual 3D boundaries and less on their representation in terms of intensity values. Consider the occluded patterns, shown in Fig. 1a. The left pattern is moving to the upper left corner with velocity $(-1, 1)$, and the right one to the lower right with velocity $(1, -1)$. Intensities are varying sinusoidally along the x -axis inside both the patterns, and are *exactly* conserved in the entire sequence. The optical flow field calculated according to oriented smoothness is shown in Fig. 1b, with constants λ and δ so chosen that the effect of isotropic component of smoothness is small. The technique suffers seriously from the so-called *aperture* problem, and fails to recover correctly the optical flow component normal to the intensity gradient. Let us explain the behavior near the horizontal edge contours. The topological gradient is oriented along the y -axis at any of these edge points. However, the intensities at these edge points are varying sinusoidally. The oriented smoothness based regularization imposes smoothness perpendicular to the gradient direction at every edge point, and incorrectly computes the y -component of the velocity field and propagates the components inside the entire pattern. Since, the direction of topological gradient matches that of intensity gradient at vertical edge points, x -component of the optical field is reasonably recovered.

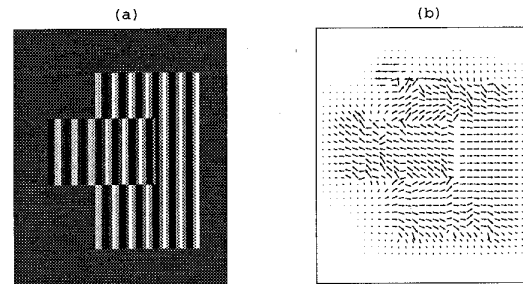


Fig. 1. Behavior of oriented smoothness based regularization. (a) Occluded patterns, the left one translating to the upper left and the right one to the lower right corner. (b) Optical flow computed by the oriented smoothness constraint.

3 FAST SCALABLE SOLUTION OF THE OPTICAL FLOW FUNCTIONAL

Optical flow estimation depends on the objective function to be minimized as well as on the accuracy, speed and robustness of the minimization technique. In this section, we show that the proposed functional leads to Euler-Lagrange equations that are second-order elliptic in nature, and their finite-difference discretization leads to a system of linear equations in optical flow components over the entire image. The stiffness matrix is positive definite and a unique flow field exists over the image domain. However, the presence of anisotropies coupled with discontinuities in the coefficients of the Euler-Lagrange equations leads to an extremely ill-conditioned stiffness matrix, and the convergence of

existing iterative methods deteriorates sharply. We present a recursive coarsening based adaptive multigrid solution technique that can generate optical flow vectors with very low computational complexity, and maintains a nearly constant high rate of convergence, practically insensitive to the amount of discontinuities present in the underlying optical flow field. This multilevel solution strategy that adapts itself according to the regularization weights, generates discontinuous optical flow field with linear complexity and very high speed.

3.1 Minimization of the optical flow functional

The optical flow field on the image plane can be obtained by minimizing the proposed objective function $\mathcal{F}_{pr}$ with respect to u and v . $\mathcal{F}_{pr}(u, v)$ is strictly convex on $[H_0^1(\Omega)]^2$. H_0^1 space is defined as

$$\begin{aligned} H_0^1(\Omega) &= \{x: \|x\|_{H_0^1(\Omega)} < \infty\} \\ \|x\|_{H_0^1(\Omega)} &= \|x\|_{L^2(\Omega)} + |x|_{H_0^1(\Omega)} \\ \|x\|_{L^2(\Omega)} &= \sqrt{\int_{\Omega} x^2 d\Omega} \quad |x|_{H_0^1(\Omega)} = \sqrt{\int_{\Omega} \nabla x \nabla x d\Omega} \end{aligned}$$

The corresponding system of Euler-Lagrange equations are second-order partial differential equations given by,

$$\begin{aligned} & (E_x^2 + \gamma(\alpha(x, y) + \beta(x, y)))u \\ & + E_x E_y v + \alpha(x, y)u_{xx} + \beta(x, y)u_{yy} = -E_x E_t \\ & (E_y^2 + \gamma(\alpha(x, y) + \beta(x, y)))v \\ & + E_x E_y u + \alpha(x, y)v_{xx} + \beta(x, y)v_{yy} = -E_y E_t \end{aligned} \quad (8)$$

with natural boundary conditions.

Now assume that the entire image domain Ω is covered by a grid which is same as the image grid. Then a five-point finite-difference approximation can be used for the problem, e.g., (8) at point (i, j) is discretized as,

$$\begin{aligned} & (E_x^2(i, j) + \gamma(\alpha(i, j) + \beta(i, j)))u(i, j) \\ & + \alpha(i, j)[-u(i, j-1) - u(i, j+1) + 2u(i, j)] \\ & + \beta(i, j)[-u(i-1, j) - u(i+1, j) + 2u(i, j)] \\ & + E_x(i, j)E_y(i, j)v(i, j) = -E_x(i, j)E_t(i, j) \end{aligned} \quad (9)$$

where, unknown $(u(i, j), v(i, j))$ is the optical flow field at point (i, j) . Thus, finite-difference discretization leads to two linear equations in unknowns $(u(i, j), v(i, j))$ at every point (i, j) . Arranging (u, v) at all image points in the lexicographic form, the system of linear equations can be expressed as,

$$Ax = b \quad (10)$$

where, A is a $2N \times 2N$ positive-definite matrix for an image with N pixels, and x is a vector consisting of optical flow components at all image points. Due to finite-difference discretization, A is highly sparse in nature, having at most five nonzeros per row, corresponding to the five-point stencil. A is the so-called *stiffness* matrix.

Any optical flow objective function should be strictly

convex with a unique minimum such that a unique solution for the optical flow field exists. In other words, the finite-difference discretization of the Euler-Lagrange equations should yield a stiffness matrix that is positive definite. In such case multilevel preconditioner based fast iterative methods can be applied for solving the resulting second-order elliptic partial differential equations. Now, consider the stiffness matrix A , arising from discretization of Euler-Lagrange equations given by (8). It can be expressed as a sum of two matrices A^{data} and A^{sm} , arising from the data constraint and smoothness constraint, respectively. A^{data} is a 2×2 block diagonal matrix with entries,

$$A_{ii}^{\text{data}} = \begin{pmatrix} E_x^2 + \gamma(\alpha + \beta) & E_x E_y \\ E_x E_y & E_y^2 + \gamma(\alpha + \beta) \end{pmatrix}$$

Entries of A^{sm} for a square image with N pixels are given by,

$$A_{ii}^{\text{sm}} = \lambda \begin{pmatrix} \alpha_{i,i-1} + \alpha_{i,i+1} + \beta_{i,i-\sqrt{N}} + \beta_{i,i+\sqrt{N}} & 0 \\ 0 & \alpha_{i,i-1} + \alpha_{i,i+1} + \beta_{i,i-\sqrt{N}} + \beta_{i,i+\sqrt{N}} \end{pmatrix}$$

$$A_{ij}^{\text{sm}} = - \begin{pmatrix} \alpha_{i,j} & 0 \\ 0 & \alpha_{i,j} \end{pmatrix}$$

for $j = i - 1$ or $j = i + 1$, and

$$A_{ij}^{\text{sm}} = - \begin{pmatrix} \beta_{i,j} & 0 \\ 0 & \beta_{i,j} \end{pmatrix}$$

for $j = i - \sqrt{N}$ or $j = i + \sqrt{N}$. Furthermore,

$$\alpha_{i,j} = 0.5 \left(\frac{|E_x(i)|}{E_x^2(i) + E_y^2(i)} + \frac{|E_x(j)|}{E_x^2(j) + E_y^2(j)} \right)$$

and

$$\beta_{i,j} = 0.5 \left(\frac{|E_y(i)|}{E_x^2(i) + E_y^2(i)} + \frac{|E_y(j)|}{E_x^2(j) + E_y^2(j)} \right)$$

Clearly, A^{data} is positive semi-definite (all Eigenvalues ≥ 0). A^{sm} is the discretization of the second-order differential operator with discontinuous and anisotropic coefficients, which is positive definite. So, A is a sum of semi-definite and positive definite matrices; thus, A is positive definite [28] and a unique discrete solution exists for the optical flow field.

3.2 Multigrid based Fast Minimization Method

The so called stiffness matrix A in (10) is of size $2N \times 2N$, with at most five entries per row. Direct methods with $O(N^3)$ computational complexity is simply impractical for such matrices. Thus local iterative methods, e.g., Gauss-Seidel, are typically employed to solve the system of equations, with $O(N)$ complexity per iteration. The number of iterations required for convergence is $O(N)$. Thus, pointwise methods need $O(N^2)$ arithmetic operations for computing optical flow from an N -pixel image and are not scalable. Irrespective of the number of processors in a multiprocessor architecture, proportionally larger solution times will be involved for larger image-size. Furthermore, the rate of

convergence is adversely affected by the presence of anisotropies coupled with discontinuities in the coefficients of the Euler-Lagrange equations given by (8). Gauss-Seidel method can slow down depending on the global contrast or frequency content of an image. A multigrid based solution strategy is employed here which maintains a size-independent high rate of convergence for all images, thereby guaranteeing $O(N)$ computational complexity.

The inherent inefficiency of pointwise iterative methods like Gauss-Seidel or Jacobi can be explained from a spatial frequency perspective. Local Fourier analysis shows that the high-frequency components of residual error $Ax(t) - b$, whose wavelengths are of the order of grid-size is reduced effectively from one iteration to the next, but the low-frequency components persist through many iterations [5]. Hence, common error norms get reduced sharply within the first few iterations and then the minimization process asymptotically slows down. While Gauss-Seidel or damped Jacobi relaxation is not efficient for reducing the entire error vector, they are quite effective for reducing high-frequencies, i.e., smoothing of error function. This suggests that multigrid methods that apply relaxations in multiscale discretizations of the continuous problem, can efficiently minimize the optical flow functional. The frequency spectrum of the error is determined by the eigenvalues of the stiffness matrix. For isotropic regularization like Horn-Schunck's, the eigenvalues of A are given by

$$\lambda_{i,j} = 4 \sin^2\left(\frac{\pi i}{2(n+1)}\right) + 4 \sin^2\left(\frac{\pi j}{2(m+1)}\right)$$

where $1 \leq i \leq n$ and $1 \leq j \leq m$ for an $n \times m$ image. Since the eigenvalues of the stiffness matrix are uniformly distributed, a hierarchy of coarse grids can be built with sizes related by a factor of 2 in each direction; and 2×2 averaging based (or 3×3 weighted averaging) restriction operators and bilinear prolongators can maintain a high convergence rate for isotropic regularization [5], [14], [20], [21]. In the case of weighted anisotropic regularization, the resulting stiffness matrix A contains almost random entries and eigenvalues are no longer uniformly distributed. The smallest eigenvalue is bounded from below by the minimum coefficient and the largest from above by the maximum one. In other words, the weighted anisotropic smoothness based regularization gives rise to *singularly perturbed equations*, where the ratio of the constant of boundedness of the bilinear form in H^1 , and the constant of V-ellipticity in H^1 can be very large. Standard multigrid techniques fail to converge fast for such systems of equations. Slowdown was reported in [8] for the multigrid implementation of oriented smoothness based regularization. In order to achieve fast convergence for anisotropic problems, Ruge introduced the concept of *strongly coupled nodes* and attempted to find interpolation operators from the condition that interpolation should prefer to proceed along strong coupling [19]. The interpolations exploit the condition that the kernel of the stiffness matrix consists of constant functions and since constant functions can be expressed easily without knowing much about the problem domain, the resulting multigrid technique was completely algebraic in nature. Such algebraic multigrid consists of two steps [16].

(1) Coarsening:

- **input:** $A_1 = A$ - the stiffness matrix.
 - **output:**
 - $\{A_l\}_{l=2}^L$, the hierarchy of coarse-level matrices of size $n_l \times n_l$, where L is the number of levels.
 - $\{P_l\}_{l=1}^{L-1}$, the family of prolongator matrices. P_l is a $n_l \times n_{l+1}$ matrix.
1. $l \leftarrow 1$
 2. construct P_l .
 3. $A_{l+1} = P_l^T A_l P_l$
 4. $l \leftarrow l+1$
 5. if n_l is large for direct solution of the corresponding system of equations, go to 1.
 6. $L \leftarrow l$.

(2) Iteration: Iteratively adjust the current flow field x .

- **input:**
 - $\{A_l\}_{l=1}^L$, $\{P_l\}_{l=1}^{L-1}$,
 - right hand side of the finest-level problem $b^1 = b$;
 - number of smoothing steps v_1 , v_2 , and initial guess x^1 .
 - **output:** the final optical flow field.
- One iteration proceeds as,
- $l \leftarrow 1$
1. Pre-smoothing: do v_1 times $x^l \leftarrow S_l(x^l, b^l)$, where $S_l(\cdot, \cdot)$ is some pointwise iterative method, e.g., Gauss-Seidel or damped Jacobi.
 2. Coarse-grid correction:
 - (a) $b^{l+1} \leftarrow P_l^T (b^l - A_l x^l)$
 - (b) solve $A_{l+1} x^{l+1} = b^{l+1}$ directly or by a recursive application of this algorithm. The exact way is defined by various multigrid cycles.
 - (c) $x^l \leftarrow x^l + P_l x^{l+1}$
 3. Post-smoothing: do v_2 times $x^l \leftarrow S_l(x^l, b^l)$.

Ruge's multigrid was based on two-level convergence estimates which do not give convergence bounds independent of the number of levels. The deterioration of convergence is indeed observed, and relatively many coarse points has to be selected, resulting in less attractive computational complexity and speed (Multilevel slowdown was observed in the hierarchical conjugate gradient method also [25]). We present below an adaptive multilevel coarsening strategy that recursively builds the restriction and prolongation operators, using the information about regularization weights given in (8).

3.2.1 Building Multilevel Prolongators Using Smoothed Aggregation

The weighted anisotropic smoothness constraint preserves discontinuities in the computed optical flow. Most of the existing restriction and prolongation operator based multilevel algorithms, e.g., those reported in [20], [21] assume the regularity of solution and so their rate of convergence is poor for weighted anisotropic regularization. Recently, a regularity-

free abstract multigrid convergence theory has been proposed in [4]. According to this theory, if the following conditions are satisfied while building the prolongators, then the algorithm would be able to maintain a high rate of convergence.

DECOMPOSITION OF UNITY. Let P_l denote the prolongator matrix between two adjacent levels l and $l+1$. The decomposition of unity requires,

$$\sum_{j=1}^{n_{l+1}} P_{ij} = 1, \quad i = 1, \dots, n_l.$$

let, $x_c \in \mathbb{R}^{n_{l+1}}$ be the coarse-level optical flow vector with all entries equal to one. The decomposition of unity enforces that $P_l x_c$ is the flow vector with all entries equal to one again. In other word, the prolongator should be built in such a way that fine-level constant flow vectors can be approximated using coarse-level constant flow vectors exactly. This requirement is motivated by the need to bound locally the error of the coarse level approximation of a fine level flow field.

SMALL ENERGY of the coarse-level flow vectors means that the "energy" of prolonged flow vectors, $\langle A_l P_l x, P_l x \rangle$ must be small. During successive coarsening, we lose the ability to approximate precisely the finest-level flow vectors—as a result, convergence may deteriorate with an increase in number of levels. This must be compensated by smoothness (low energy) of coarse-level flow vectors.

SHAPE OF SUPPORTS of basis functions must be such that the support of any basis function (one column of P_l) contains strongly coupled nodes only. For example, strongly anisotropic equations, i.e., $\alpha(x, y)\beta(x, y) \approx 0$ can be solved in such a way that coarsening will be done in the direction of anisotropy only (so-called semi-coarsening). Algebraically, the anisotropy is reflected in terms of regularization weights (α, β) which appear in the coefficients of the stiffness matrix given in (10).

SMALL FILL-IN. The number of nonzero entries in coarse-level problem $P_l^T A_l P_l$ should be small. This is important from both practical as well as theoretical points of view. Large number of nonzeros reduces the performance of the iterative process (speed and memory) and less attractive computational complexity, since in sparse matrix algebra, number of operations is of the order of the number of nonzeros in the matrix. From the theoretical point of view, the bounded number of nonzeros per column in the coarse-level matrices is closely related to the bounded number of intersections of supports of basis functions. This so-called 'bounded intersection property' is important for the localization of approximation estimates [27].

Now we present a coarsening algorithm that utilizes the information about the smoothness weights α, β in generation of prolongators $\{P_l\}$ to satisfy the above requirements. For the clarity of presentation, we first elaborate the algorithm for one flow component/node and then mention a simple extension for two flow components/node. This recursive algorithm is based on combination of two techniques:

- 1) *unknown aggregation* to generate auxiliary prolongators, followed by
- 2) *smoothing* of auxiliary prolongators. Unknown aggregation generates prolongators that satisfy the requirements of decomposition of unity, small size of coarse-level matrices, and the shape of supports. However, these auxiliary prolongators do not satisfy the requirement of small energy of coarse-level flow vectors. Smoothing is performed to ensure that energy contained in the coarse-level flow field is small.

The unknown aggregation technique works as follows. First, the disjoint covering of the set of nodes $\{C_i\}_{i=1}^m$ is generated, where each set C_i contains strongly coupled nodes only. The information about the coupling of nodes (i.e., α s and β s) can be obtained from node entries of the stiffness matrix in (10). The strongly coupled neighborhood, N_i^l of node i is defined as,

$$N_i^l = \{j: |a_{ij}| \geq \epsilon \sqrt{a_{ii}a_{jj}}\}.$$

If the stiffness matrix is diagonally dominant, it is more appropriate to use,

$$N_i^l = \{j: |a_{ij}| \geq \epsilon \sqrt{q_i q_j}\}$$

where

$$q_i = \max_{k \neq i} |a_{ik}|, \quad q_j = \max_{k \neq j} |a_{jk}|,$$

and $\epsilon \in (0, 1)$. These sets are small and contain only a few nodes. If $\alpha = \beta = \text{constant}$, C_i s contain four-connected neighbors of the node i . The auxiliary prolongators are built as,

$$P_{ij} = \begin{cases} 1 & \text{if } i \in C_j \\ 0 & \text{otherwise} \end{cases}$$

P is very sparse and contains one nonzero entry per row. The computational steps are detailed below:

Auxiliary prolongators: Generate the auxiliary prolongation operators, $\{\hat{P}_l\}$.

- **input**
 - A_l , the coarse-level matrix of order n_l on level l
 - **output**
 - $\hat{P}_l$, the auxiliary prolongator matrix of size $n_l \times n_{l+1}$
- ```

R ← {1, ..., n_l}, j ← 0.
for i := 1 to n_l do
 if N_i^l ⊂ R then
 j ← j + 1,
 C_j ← N_i^l,
 R ← R \ C_j
 end if
end for

for i := 1 to n_l do
 if i ∈ R then
 find C_k: N_i^l ∩ C_k ≠ ∅,
```

```

 Ck ← Ck ∪ {i}
 end if
end for
set nt+1 ← j

```

Generate  $\hat{P}_l: \hat{P}_l: \mathbb{R}^{n_{t+1}} \rightarrow \mathbb{R}^{n_t}$  by

$$(\hat{P}_l)_{ij} = \begin{cases} 1 & \text{if } i \in C_j \\ 0 & \text{otherwise} \end{cases}$$

Final prolongators can be obtained by applying a damped Jacobi smoother to the auxiliary prolongators. The computational steps are presented below:

Smooth the auxiliary prolongators  $\{\hat{P}_l\}$  to obtain final prolongators  $\{P_l\}$ .

- **input**  
-  $A_\nu \{\hat{P}_l\}$
- **output**  
-  $P_l$ , the improved prolongator matrix
- Generate the filtered matrix  $A_l^f = \{b_{ij}\}_{i,j=1}^{n_l}$  from the matrix  $A_l = \{a_{ij}\}_{i,j=1}^{n_l}$ :  
- offdiagonal entries, i.e.,  $i \neq j$ :

$$b_{ij} = \begin{cases} a_{ij} & j \in N_i^l \\ 0 & \text{otherswise} \end{cases}$$

- diagonal entries, i.e.,  $i = j$ :

$$b_{ij} = a_{ii} - \sum_{j=1, j \neq i}^{n_l} (a_{ij} - b_{ij})$$

- Smooth the auxiliary prolongators:  
- Define the smoother

$$S_l = I - w D^{-1} A_l^f$$

where  $D = \text{diag}(A_l^f)$ , and compute the new prolongator  $P_l$  by

$$P_l = S_l \hat{P}_l$$

For a large class of problems, we have experimented with, reasonable choices of parameters are

$$\varepsilon = 0.2, \quad w = \frac{4}{3}.$$

The coarsening process elaborated above is for the case of one component/node. We can easily generalize the scalar algorithm for two flow components/node using the so-called *block* approach. The idea is to compute with matrices of order two instead of computing with a single number at each node, which describes the communication between neighboring nodal points. Let  $df(k)$  denote the list of flow components associated with node  $k$ . The matrix describing the communication between nodes  $k$  and  $l$  is the selection

$$A_{kl} = A(df(k), df(l)).$$

In order to generalize the algorithm, two minor changes are required. The definition of the set of strongly coupled nodes can be generalized as

$$N_i^l = \left\{ j: \rho(A_{ij}^l) \geq \varepsilon \sqrt{\rho(A_{ii}^l) \rho(A_{jj}^l)} \right\};$$

and for diagonally dominant matrices as,

$$N_i^l = \left\{ j: \rho(A_{ij}^l) \geq \varepsilon \sqrt{q_i q_j} \right\}$$

where  $q_i = \max_{k \neq i} \rho(A_{ik})$ .  $\rho(\cdot)$  denotes the spectral bound of square matrix. Also, the 0 and 1's are replaced by zero and identity matrices of order two, respectively.

## 4 EXPERIMENTAL RESULTS

Numerous experiments have been conducted with synthetic and real image sequences to demonstrate the efficacy of the proposed optical flow estimation technique. Some of these are reported here. All experiments are conducted on a IBM RS6000/360 workstation. Detected flows are pictorially presented for the proposed, Anandan's and Zheng-Blostein's techniques. For synthetic image sequences, where optical flow fields are known a priori, errors are calculated for moving regions and background regions separately and then averaged. These errors are calculated in degree, as suggested in [2]:

$$\text{error} = \cos^{-1} \left( \frac{1 + u_c u_d + v_c v_d}{\sqrt{(1 + u_c^2 + v_c^2)(1 + u_d^2 + v_d^2)}} \right)$$

where  $(u_c, v_c)$  and  $(u_d, v_d)$  are respectively the correct and detected flow vectors at an image point. Partial derivatives are computed using four-point central difference with  $\frac{1}{12}[-1, 8, 0, -8, 1]$ . Real sequences are presmoothed with a spatiotemporal Gaussian with standard deviation  $\sigma = 1$  in every direction to reduce aliasing. No smoothing is done in the case of synthetic sequences.  $k_1 = 150$ ,  $k_2 = 1$  and  $k_3 = 0$  are used in Anandan's optical flow technique. Since the maximum displacement is less than two pixels/frame in synthetic sequences, a  $5 \times 5$  search window has been used. Laplacian pyramid has been employed for real sequences.  $k_1 = 180$ ,  $k_2 = 1000$  and  $k_3 = 0$  are used in Zheng-Blostein's flow algorithm, as suggested in [30]. A  $7 \times 7$  search window has been used.

Figs. 2a-d show frames of four synthetic image sequences, used in our numerical experiments. In Fig. 2a a bright pattern is moving to upper left corner with inter-frame displacement of  $(-1, 1)$  against the stationary background. The vertical component of the flow field is perpendicular to the intensity variation in this sequence. The right pattern is moving to the left with an inter-frame displacement of  $(-1, 0)$  and the left one is moving upward with displacement  $(0, 1)$  in Fig. 2b. Thus, the underlying flow field is either in horizontal or in vertical direction, perpendicular to intensity variations in the corresponding pattern.

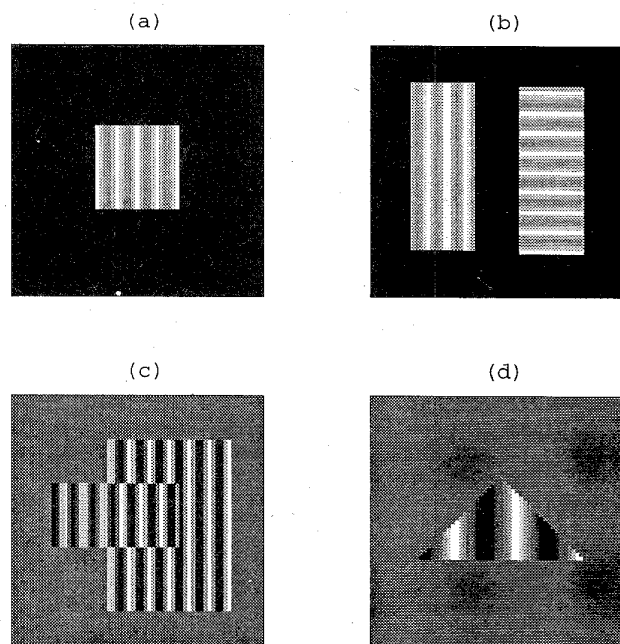


Fig. 2. Synthetic image frames. (a) Diagonally translating pattern with intensity variation in only  $x$ -direction. (b) Two plaid patterns moving horizontally and vertically. (c) Two partially occluded blocks moving toward the upper left and lower right corners. (d) The background is rotating and the triangular pattern is translating upward.

The image sequence in Fig. 2c shows occluded patterns. The left pattern is moving with velocity  $(-1, 1)$  and the right one with  $(1, -1)$ . No intensity gradient information is present in vertical direction inside both patterns. The background is rotating counter-clockwise with respect to the center with an angular velocity of  $1 \text{ deg/frame}$  in Fig. 2d, and the triangular pattern is translating upward with a velocity of  $1 \text{ pixel/frame}$ . Flow fields estimated using the proposed, Anandan's and Zheng-Blostein's techniques are subsampled and plotted in Figs. 3a-d, Figs. 4a-d, and Figs. 5a-d, respectively. The numerical errors are reported in Tables 1-4, respectively.

Experiments with synthetic image sequences clearly demonstrate the ability of the weighted anisotropic regularization to preserve discontinuities in the computed flow in different directions as well as to compute flows accurately in intensity patterns varying in one direction only. Results are reported with  $\lambda = 35.0$ ,  $\gamma = 0.0$  for sequences shown in Figs. 2a-c. Since strong spatial information is available at all image points in Fig. 2d, a smaller  $\lambda = 10$  has been used. Anandan's technique worked fairly well for Fig. 2d where strong texture is available at all image points. For Figs. 2a and c, it could only compute the optical flow component in the direction of intensity gradient. Flow discontinuities, in general, are not preserved as good as in the proposed technique. In Fig. 2b, flow could be determined only near boundary points, since no matching error information is available inside both objects. Zheng-Blostein's error-weighted regularization was able to recover the flow fields better than Anandan's technique. But it too failed to recover the flow component perpendicular to the intensity gradient. Note

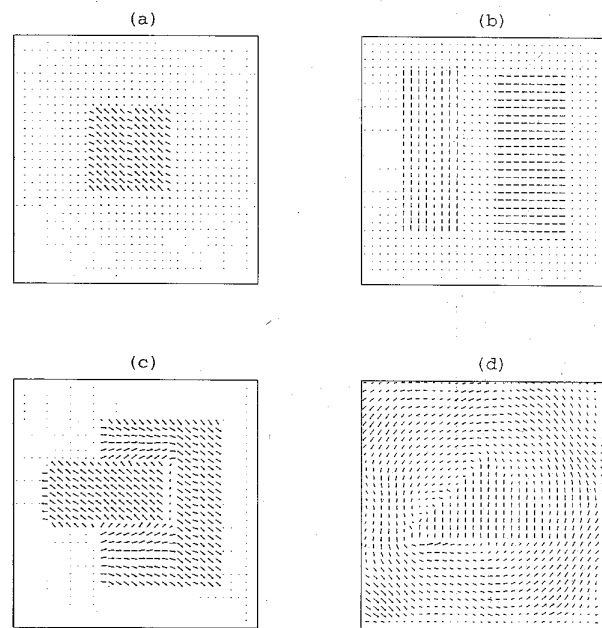


Fig. 3. Optical flow computation using weighted anisotropic regularization.

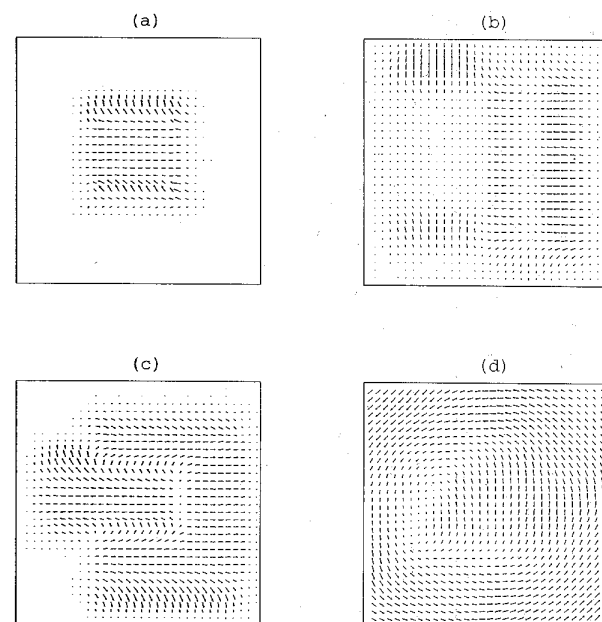


Fig. 4. Optical flow computation using Anandan's technique.

that in these synthetic sequences the interframe displacement is less than approximately two pixels. Error-weighted regularization seems to work better for larger displacement as well as for images with textures in both background and object points. The proposed technique can be easily implemented hierarchically for handling large displacement without increasing the computation too much.

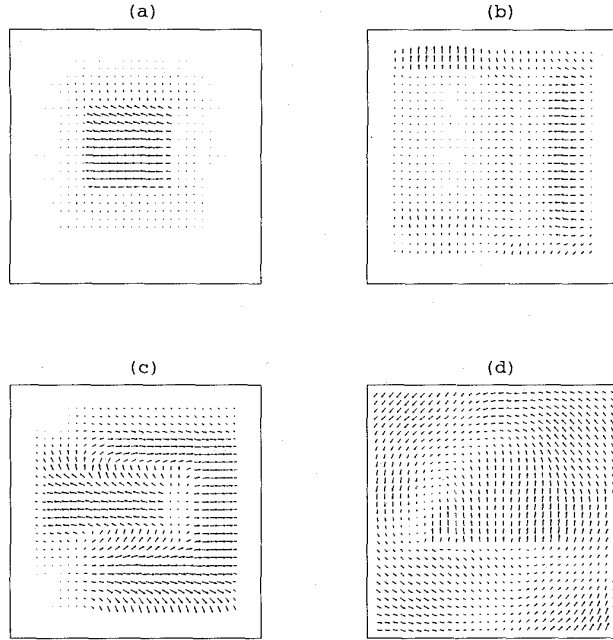


Fig. 5. Optical flow computation using error-weighted regularization.

The proposed technique outperforms both Anandan's and Zheng-Blostein's techniques also in terms of speed and robustness, in some cases by an order of magnitude. For all the  $128 \times 128$  synthetic sequences, the proposed technique takes around 20 sec. of user time, when iterative process is terminated for relative residue dropping below  $10^{-5}$ . Anandan's technique was terminated when the scaled  $L_2$  (r.m.s.) difference between flow vectors in successive iterations dropped below  $10^{-3}$ . It took approximately 3.2 mins., 5.5 mins., 10 mins. for sequences shown in Figs. 2a, 2b, and 2c, respectively. It took more than 500 iterations to drop the error below  $10^{-2}$  in Fig. 2d. Note that such stopping condition ensures that the solution vector at any point changes no more than approximately 10% of the pixel spacing between successive iterations. Clearly, Gauss-Seidel is not suitable to be used for robust computation of optical flow. For the same stopping condition, Zheng-Blostein's technique is even slower. In the initial phase of Gauss-Seidel relaxation, errors are reduced sharply for the error-weighted regularization, but in the later phase the convergence slows down asymptotically.

Figs. 6a-d shows the first and last frames of four real sequences. The optical flow is estimated in the middle frame. In the Hamburg Taxi sequence, a car in the lower left is moving to right, a van in the lower right backing up, and a taxi turning the corner. In the infrared Car sequence, the camera is moving horizontally and the car is turning the curb. The turntable is rotating counter-clockwise in the Rotating Cube sequence. The turntable is displaced more on the image plane than others due to perspective. In the Nasa sequence, the camera is moving along its line of sight toward the soda can near the center of the image. Thus, the motion

TABLE 1  
ERRORS IN OPTICAL FLOW FOR THE SEQUENCE  
SHOWN IN FIG. 2A

|            | Anandan's | Zheng-Blostein's | Proposed |
|------------|-----------|------------------|----------|
| Object     | 44.02     | 34.90            | 4.82     |
| Background | 4.71      | 2.08             | 5.17     |
| Average    | 24.36     | 18.49            | 4.99     |

TABLE 2  
ERRORS IN OPTICAL FLOW FOR THE SEQUENCE  
SHOWN IN FIG. 2B

|            | Anandan's | Zheng-Blostein's | Proposed |
|------------|-----------|------------------|----------|
| Object     | 48.32     | 33.86            | 0.79     |
| Background | 12.83     | 6.98             | 6.28     |
| Average    | 30.57     | 20.42            | 3.54     |

TABLE 3  
ERRORS IN OPTICAL FLOW FOR THE SEQUENCE  
SHOWN IN FIG. 2C

|            | Anandan's | Zheng-Blostein's | Proposed |
|------------|-----------|------------------|----------|
| Object     | 41.32     | 35.72            | 16.44    |
| Background | 15.56     | 7.96             | 1.72     |
| Average    | 28.44     | 21.84            | 9.08     |

TABLE 4  
ERRORS IN OPTICAL FLOW FOR THE SEQUENCE  
SHOWN IN FIG. 2D

| Anandan's | Zheng-Blostein's | Proposed |
|-----------|------------------|----------|
| 14.64     | 18.49            | 13.13    |

is primarily dilational. Results obtained using the weighted anisotropic smoothness based optical flow technique are shown in Figs. 7a-d. Results obtained using Anandan's and Zheng-Blostein's techniques are presented in Figs. 8a-d and Figs. 9a-d, respectively.  $\lambda$  and  $\gamma$  are chosen to be 35.0 and 0.01%, respectively for the real sequences. For error tolerance of  $10^{-3}$ , the proposed technique took approximately 50 sec., 30 sec., 55 sec., and 80 sec. for flow computation in the Hamburg Taxi, Car, Cube and Nasa sequences. For a stopping error of 0.1, Anandan's technique took approximately 10 mins., 5.5 mins., 17 mins., 15 mins. in corresponding sequences. Zheng-Blostein's technique took longer time to converge. For results obtained by other well-known techniques, see [2].

All these results demonstrate the effectiveness and speed of the proposed discontinuous optical flow estimation technique.

#### 4.1 Scalability of the Proposed Algorithm

The speed of the entire optical flow algorithm is of immense importance in real-world applications. Numerical solution of the Euler-Lagrange equations is the most time consuming part, and time complexity of the solution grows as high as the square of problem size, e.g., in popular Gauss-Seidel iterative method. As a result, in spite of a large number of available processors in the optimistic case, solution times will be larger for larger images. Fast and scalable algorithms are required to ensure execution times to be only linearly dependent on image size in a serial computing environment. Scalable algorithms involving matrix-vector operations are perfectly parallelizable on vector processors,

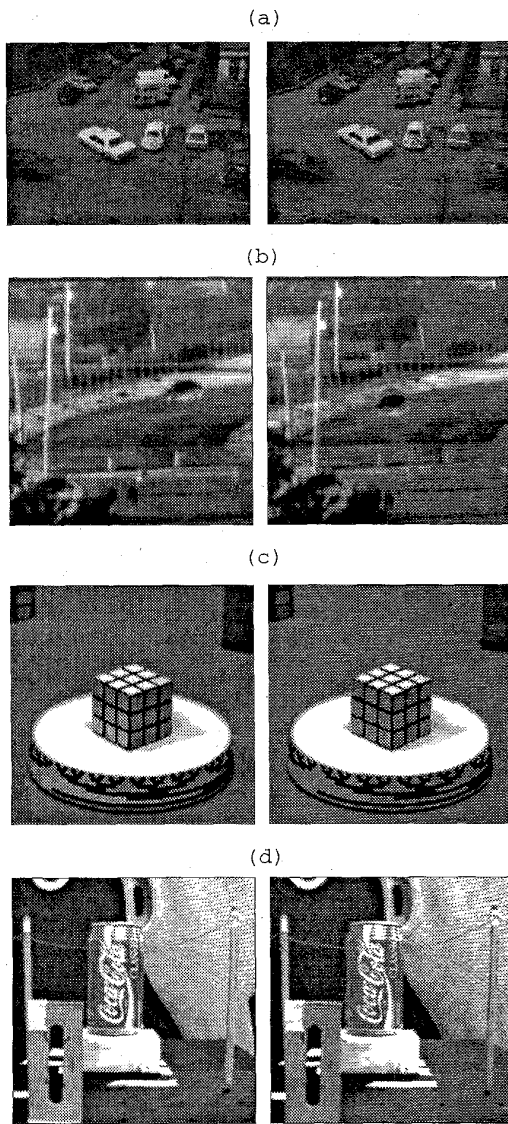


Fig. 6. Real image sequences. (a) Hamburg Taxi sequence. (b) Infrared Car sequence. (c) Rotating cube sequence. (d) Nasa sequence.

and thus image size-independent solution times can be obtained. Fig. 10 shows first and last frames of a  $256 \times 512$  synthetic image sequence. The image frames are subsampled (uniformly in both directions) to create smaller-size images. Optical flows are computed according to the weighted anisotropic regularization for images of different size and user times are plotted in Fig. 11 for error tolerance of  $10^{-5}$ . The linear nature of the plot confirms the scalability of the proposed discontinuous optical flow technique. Execution times are plotted in Fig. 12 for some of the real sequences shown in Fig. 6. Corresponding times are also shown when Gauss-Seidel relaxation is performed to generate flow field with the same maximum tolerance. Our multilevel solution strategy (with  $V(1, 1)$  cycle) took at most five iterations to converge for all images reported here, whereas the block Gauss-Seidel (more efficient than usual) method took as high as 1,800

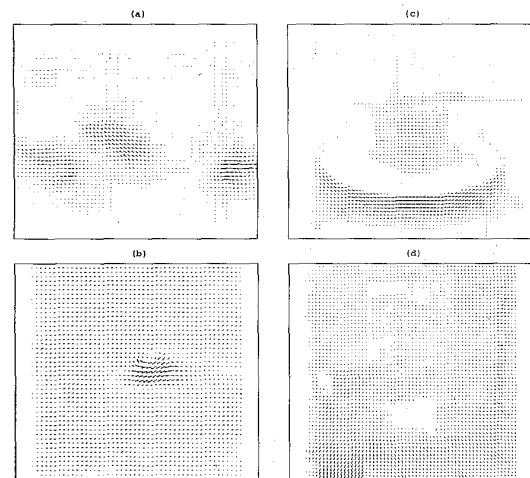


Fig. 7. Optical flow computation for real sequences using the proposed technique. Flow field in (a) Hamburg Taxi sequence. (b) Infrared Car sequence. (c) Rotating cube sequence. (d) Nasa sequence.

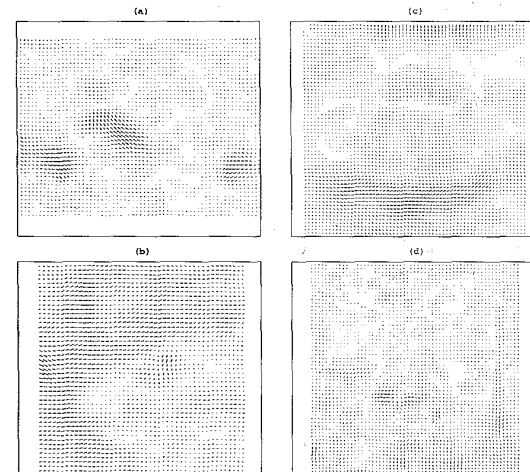


Fig. 8. Optical flow computation for real sequences using Anandan's technique. Flow field in (a) Hamburg Taxi sequence. (b) Infrared Car sequence. (c) Rotating cube sequence. (d) Nasa sequence.

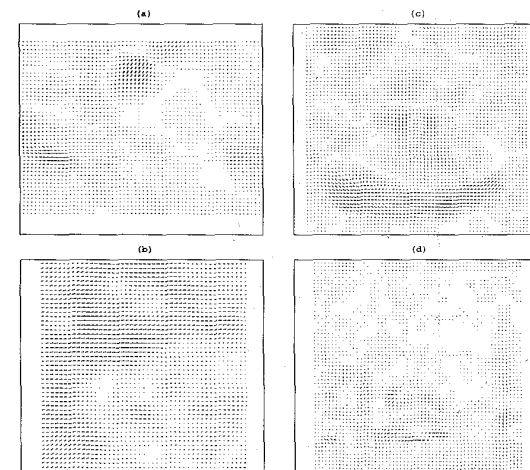


Fig. 9. Optical flow computation for real sequences using Zheng-Blostein's technique. Flow field in (a) Hamburg Taxi sequence. (b) Infrared Car sequence. (c) Rotating cube sequence. (d) Nasa sequence.

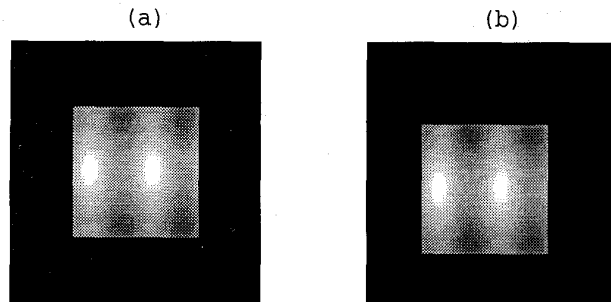


Figure 10. Synthetic image sequence used to test the scalability of the proposed optical flow technique.

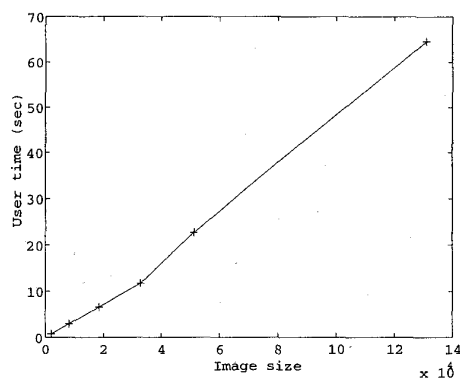


Fig. 11. Scalability of the proposed optical flow technique.

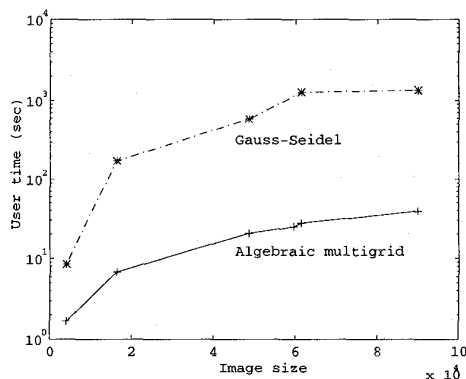


Fig. 12. Comparison of execution speeds between Gauss-Seidel and the smoothed aggregation based multilevel solution technique for real sequences.

iterations to converge. Cost of each multigrid iteration is approximately five times that of the Gauss-Seidel's. The algorithmic complexity (per iteration) is 1.15–1.33 for the proposed multilevel solution technique, compared to 1.33 for two-to-one conventional coarsening, and 1.9–2.5 for [19]. These results clearly establish the speed and scalability of our multilevel computational approach for discontinuous optical flow estimation.

## 5 CONCLUSIONS

A weighted anisotropic smoothness based regularization technique is proposed in this paper for estimating discontinuous optical flows from two-frame sequences. Intensity gradient-weighted penalization of the magnitude of the solution is suggested for preventing the propagation of flow field in low-texture background regions. An adaptive multilevel implementation is presented for efficiently computing discontinuous optical flow field. Scalability and speed of our optical flow technique are demonstrated experimentally for both synthetic and real image sequences. Future research will include investigation of intelligent schemes for adapting the constant of smoothness based on the present solution, to account for the inherent nonlinearity in the optical flow problem in the framework of linear systems of equations.

## ACKNOWLEDGMENT

The authors would like to thank Jan Mandel of the University of Colorado at Denver for his comments on an earlier draft of this manuscript, and the University of Western Ontario, Canada for providing FTP access to real image sequences and implementations of Anandan's and Nagel-Enkelmann's optical flow techniques.

This research is supported by the National Science Foundation grants no. ASC-9217394, ASC-9404734.

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